Problem 1.47

(a) Write an expression for the volume charge density $\rho(\mathbf{r})$ of a point charge q at $\mathbf{r}'$. Make sure that the volume integral of ρ equals q.

(b) What is the volume charge density of an electric dipole, consisting of a point charge $-q$ at the origin and a point charge $+q$ at $\mathbf{a}$?

(c) What is the volume charge density (in spherical coordinates) of a uniform, infinitesimally thin spherical shell of radius R and total charge Q, centered at the origin? [*Beware:* The integral over all space must equal Q.]

Solution

Part (a)

The volume charge density for a point charge q at $\mathbf{r}'$ is

$$\rho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}').$$

Part (b)

The volume charge density for a point charge $-q$ at the origin and a point charge $+q$ at $\mathbf{a}$ is

$$\rho(\mathbf{x}) = -q\delta(\mathbf{x}) + q\delta(\mathbf{x} - \mathbf{a}).$$

Part (c)

Since the spherical shell exists entirely at $r = R$, only the delta function $\delta(r - R)$ is necessary to describe the charge density.

$$\rho(\mathbf{x}) = A\delta(r - R)$$

A is a normalization constant: Determine it by requiring the integral over a volume containing the shell to be Q.

$$\begin{aligned} Q = \iiint \rho(\mathbf{x})\, d\mathbf{x} &= \int_0^\pi \int_0^{2\pi} \int_0^\infty A\delta(r - R)(r^2 \sin\theta\, dr\, d\phi\, d\theta) \\ &= A\left(\int_0^\pi \sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right)\left[\int_0^\infty \delta(r - R) r^2\, dr\right] \\ &= A(2)(2\pi)(R^2) \end{aligned}$$

Therefore,

$$A = \frac{Q}{4\pi R^2},$$

and the charge density is

$$\rho(\mathbf{x}) = \frac{Q}{4\pi R^2}\delta(r - R).$$